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On square roots of the Haar state on compact quantum groups

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ABSTRACT

The paper is concerned with the extension of the classical study of probability measures on a compact group which are square roots of the Haar measure, due to Diaconis and Shahshahani, to the context of compact quantum groups. We provide a simple characterisation for compact quantum groups which admit no non-trivial square roots of the Haar state in terms of their corepresentation theory. In particular it is shown that such compact quantum groups are necessarily of Kac type and their subalgebras generated by the coefficients of a fixed two-dimensional irreducible corepresentation are isomorphic (as finite quantum groups) to the algebra of functions on the group of unit quaternions. An example of a quantum group whose Haar state admits no nontrivial square root and which is neither commutative nor cocommutative is given.

1. Introduction

If G is a (locally) compact group, then the space $\mathrm{M}(G)$ of all bounded regular measures on G is equipped with a natural product $\star$ afforded by the convolution, making $\mathrm{M}(G)$ a Banach algebra. In particular a convolution of two probability measures remains a probability measure. Convolution equations in $\mathrm{M}(G)$ have often natural interpretations—for example the Haar measure μ_G on compact G can be described as a unique probability measure in $\mathrm{M}(G)$ such that $\mu_G\star\mu = \mu\star\mu_G = \mu_G$ for all $\mu \in \mathrm{Prob}(G)$, idempotents in $\mathrm{Prob}(G)$ can be characterised as Haar measures on compact subgroups of G (Kawada–Itô Theorem), and so on. In [3] Diaconis and Shahshahani showed that the Haar measure of a separable compact topological group G does not admit a non-trivial square root, i.e., a probability $\nu \neq \mu_G$ with $\nu \star \nu = \mu_G$, if and only if G is abelian or of the form $H \times E$, where H is the eight element group of unit quaternions and E a product of two element groups.

If (A, Δ) is a compact quantum group in the sense of Woronowicz [28], then the quantum counterpart of the set of the probability measures on the group is given by the state space of A, $S(\mathsf{A})$. It is again equipped with a natural convolution operation and it makes sense to ask for the solutions of analogous equations as those listed above. Thus for example the Haar state can be described as a unique state h on A such that for all $\rho \in S(\mathsf{A})$ there is $h \star \rho = \rho \star h = h$, but on the other hand there may exist idempotent states on A which do not arise as Haar states on compact quantum subgroups of A (see [5,7,18]).

In this paper we investigate the quantum counterpart of the question studied by Diaconis and Shahshahani—which compact quantum groups belong to the *quantum DS-family*, i.e., have the property that their Haar state does not admit a non-trivial square root? The proof of the characterisation in [3] consists of three main steps: first they show that the

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existence of the nontrivial square roots of the Haar measure on G is equivalent to the existence of a non-zero bounded real nilpotent measure on G, then deduce from this that if G admits no such nilpotent measures then it must be hamiltonian (i.e., all its closed subgroups are normal) and finally classify compact separable hamiltonian groups and use this classification to complete the proof.

Here we first show that the existence of the nontrivial square roots of the Haar state is equivalent to the fact that the dual discrete algebraic quantum group $(\hat{\mathcal{A}}, \hat{\Delta})$ contains a hermitian non-zero nilpotent element, cf. Theorem 3.8 (the proof in our context is similar to that of [3], but substantially more technical). Motivated by our work on idempotent states, we propose a definition of a *hamiltonian compact quantum group* as the one on which all idempotent states are central and show that if (A, Δ) belongs to the quantum *DS*-family, then it is hamiltonian, cf. Proposition 3.12 (in particular, all compact quantum subgroups of (A, Δ) are normal). As the classification of hamiltonian compact quantum groups is currently beyond our reach, to continue the investigation we need to provide another strategy, based on the corepresentation theory. It turns out that the quantum *DS*-family consists exactly of those compact quantum groups which have only one- and two-dimensional irreducible corepresentations and satisfying a certain additional condition on the linear functionals coming from their two-dimensional corepresentations, cf. Theorem 4.1. This allows us to deduce that if (A, Δ) is in the quantum *DS*-family, then it is necessarily of Kac type and its subalgebras generated by the coefficients of a fixed two-dimensional irreducible corepresentation are isomorphic to the algebra of functions on the group of unit quaternions. That result and some further observations on the interaction between two-dimensional and one-dimensional corepresentations are used to provide an explicit example of a compact quantum group in the quantum *DS*-family which is neither commutative nor cocommutative.

The detailed plan of the paper is as follows: Section 2 contains all the preliminary facts and terminology related to compact (and discrete) quantum groups and their corepresentations. Here we also introduce the fundamental notion of the square root of the Haar state and characterise commutative and cocommutative elements of the quantum *DS*-family. Section 3 is devoted to establishing the equivalence between the existence of nontrivial square roots and non-zero bounded hermitian nilpotent functionals and discuss hamiltonian compact quantum groups. In Section 4 the corepresentation theory starts to play a prominent role, providing a means to characterise the quantum *DS*-family. This is used in the following section to show that the members of the quantum *DS*-family are necessarily compact quantum groups of Kac type and to obtain a description of their 'local' structure. Different parts of this 'local' structure are combined in Section 6 to construct an example of a compact quantum group which admits no non-trivial square root of the Haar state and yet is neither commutative nor cocommutative. In that section we also state an open problem related to the possible 'degree of complication' of two-dimensional irreducible corepresentations of a quantum group in the quantum *DS*-family.

2. Preliminaries

The symbol $\otimes$ will denote the spatial tensor product of C^*-algebras and $\odot$ the algebraic tensor product, we use $\mathrm{Lin}F$ for the linear span of a set F in a vector space and $\overline{\mathrm{Lin}}F$ for the closed linear span of a set F in a Banach space.

2.1. Compact quantum groups

The notion of compact quantum groups has been introduced in [26]. Here we adopt the definition from [28] (Definition 2.1 of that paper).

Definition 2.1. A *C^*-bialgebra* (a compact quantum semigroup) is a pair (A, Δ), where A is a unital C^*-algebra, $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ is a unital, $*$-homomorphic map which is coassociative, i.e.,

$$(\Delta \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta = (\mathrm{id}_{\mathsf{A}} \otimes \Delta) \circ \Delta.$$

If the quantum cancellation properties

$$\overline{\mathrm{Lin}}((1 \otimes \mathsf{A})\Delta(\mathsf{A})) = \overline{\mathrm{Lin}}((\mathsf{A} \otimes 1)\Delta(\mathsf{A})) = \mathsf{A} \otimes \mathsf{A},$$

are satisfied, then the pair (A, Δ) is called a *compact quantum group*.

The map Δ is called the *coproduct* of A, it induces the convolution product

$$\lambda \star \mu := (\lambda \otimes \mu) \circ \Delta, \quad \lambda, \mu \in \mathsf{A}^*.$$

When the coproduct is clear from the context we just speak of a compact quantum group A.

The following fact is of the fundamental importance for this paper, cf. [28, Theorem 2.3].

Proposition 2.2. *Let (A, Δ) be a compact quantum group. There exists a unique state $h \in \mathsf{A}^*$ (called the* Haar state *of A) such that for all $a \in \mathsf{A}$*

$$(h \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta(a) = (\mathrm{id}_{\mathsf{A}} \otimes h) \circ \Delta(a) = h(a)1.$$

This naturally leads to the next definition introducing the main object of interest for the rest of the paper.

Definition 2.3. A state ϕ on a compact quantum group A is called a square root of the Haar state if

$$\phi \star \phi = h.$$

It is said to be non-trivial if $\phi \neq h$.

In general, the Haar state of a compact quantum group need not be faithful. But one can always divide by the nullspace of the Haar state to produce a compact quantum group with faithful Haar state, usually called the reduced version of the original quantum group, cf. [2]. This construction allows us to reduce our study to compact quantum groups with faithful Haar states, see Lemma 3.6.

2.2. Quantum subgroups

The notion of a quantum subgroup was introduced by Kac [9] in the setting of finite ring groups and by Podleś [17] for matrix pseudo-groups. In some contexts related to quantum subgroups it is necessary to distinguish between the reduced and universal versions of the compact quantum groups in question (or consider coamenable compact quantum groups, for which the two versions coincide), but it will not be important here.

Definition 2.4. A compact quantum group $(\mathsf{B}, \Delta_{\mathsf{B}})$ is said to be a quantum subgroup of a compact quantum group $(\mathsf{A}, \Delta_{\mathsf{A}})$ if there exists a surjective compact quantum group morphism $\pi : \mathsf{A} \to \mathsf{B}$, i.e., a surjective unital $*$-homomorphism $\pi : \mathsf{A} \to \mathsf{B}$ such that

$$\Delta_{\mathsf{B}} \circ \pi = (\pi \otimes \pi) \circ \Delta_{\mathsf{A}}.$$

A quantum subgroup B of A with Haar state h_B is called *normal* if the images of the conditional expectations

$$E_{\mathsf{A}/\mathsf{B}} = \big(\mathrm{id} \otimes (h_{\mathsf{B}} \circ \pi)\big) \circ \Delta_{\mathsf{A}},$$
$$E_{\mathsf{B}\backslash\mathsf{A}} = \big((h_{\mathsf{B}} \circ \pi) \otimes \mathrm{id}\big) \circ \Delta_{\mathsf{A}},$$

coincide, cf. [24, Proposition 2.1 and Definition 2.2]. Note that the images of the conditional expectations above can be thought of as the algebras of functions constant respectively on the right and left 'cosets' of the quantum subgroup B.

2.3. Corepresentations

An element $u = (u_{k\ell})_{1\leq k,\ell\leq n} \in M_n(\mathsf{A})$ is called an *n-dimensional corepresentation of* (A, Δ) if for all $k, \ell = 1, \ldots, n$ we have $\Delta(u_{k\ell}) = \sum_{j=1}^n u_{kj} \otimes u_{j\ell}$. All corepresentations considered in this paper are supposed to be finite-dimensional. A corepresentation u is said to be *non-degenerate*, if u is invertible, *unitary*, if u is unitary, and *irreducible*, if the only matrices $T \in M_n(\mathbb{C})$ with $Tu = uT$ are multiples of the identity matrix. Two corepresentations $u, v \in M_n(\mathsf{A})$ are called *equivalent*, if there exists an invertible matrix $U \in M_n(\mathbb{C})$ such that $Uu = vU$.

An important feature of compact quantum groups is the existence of the dense $*$-subalgebra $\mathcal{A}$ (the algebra of the *smooth* elements of A), which is in fact a Hopf $*$-algebra with the coproduct $\Delta|_{\mathcal{A}}$ – so for example $\Delta : \mathcal{A} \to \mathcal{A} \odot \mathcal{A}$. If we fix a complete family $(u^{(s)})_{s\in\mathcal{I}}$ of mutually inequivalent irreducible unitary corepresentations of (A, Δ), then $\{u^{(s)}_{k\ell}; s \in \mathcal{I}, 1 \leq k, \ell \leq n_s\}$ (where n_s denotes the dimension of $u^{(s)}$) is a linear basis of $\mathcal{A}$, cf. [28, Proposition 5.1]. We shall reserve the index $s = \emptyset$ for the trivial representation $u^{\emptyset} = \mathbf{1}$.

Set $V_s = \mathrm{span}\,\{u^{(s)}_{k\ell}; 1 \leq k, \ell \leq n_s\}$ for $s \in \mathcal{A}$. By [28, Proposition 5.2], there exists a unique irreducible unitary corepresentation $u^{(s^c)}$, called the *contragredient* representation of $u^{(s)}$, such that $V_s^* = V_{s^c}$. Clearly $(s^c)^c = s$.

The matrix elements of the irreducible unitary corepresentations satisfy the famous Peter–Weyl orthogonality relations

$$h\left(\left(u^{(s)}_{ij}\right)^* u^{(t)}_{k\ell}\right) = \frac{\delta_{st}\delta_{j\ell}\overline{f\left(\left(u^{(s)}_{ki}\right)^*\right)}}{D_s} \tag{2.1}$$

where $f : \mathcal{A} \to \mathbb{C}$ denotes the so-called Woronowicz character and

$$D_s = \sum_{\ell=1}^{n_s} f\left(u^{(s)}_{\ell\ell}\right) = \sum_{\ell=1}^{n_s} \overline{f\left(\left(u^{(s)}_{\ell\ell}\right)^*\right)}$$

is the quantum dimension of $u^{(s)}$, cf. [26, Theorem 5.7.4]. Note that unitarity implies that the matrix

$$f\left(\left(u^{(s)}_{k\ell}\right)^*\right) \in M_{n_s}(\mathbb{C})$$

is invertible, with inverse $\big(f(u^{(s)}_{k\ell})\big) \in M_{n_s}(\mathbb{C})$, cf. [26, Equations (5.18), (5.24)]

We will say that a linear functional $\phi : \mathsf{A} \to \mathbb{C}$ has *finite support*, if

$$\varphi|_{V_s} = 0$$

for all but finitely many $s \in \mathcal{I}$. The Haar state has finite support, since it vanishes on all irreducible unitary representations except the trivial one.

Lemma 2.5. *A continuous linear functional $\phi : \mathsf{A} \to \mathbb{C}$ has finite support if and only if it admits a smooth density w.r.t. to the Haar state, i.e., if there exists $x \in \mathcal{A}$ such that*

$$\phi(a) = h(xa), \quad \text{for all } a \in \mathsf{A}.$$

The density x is uniquely determined by ϕ.

Proof. Assume such a density $x \in \mathcal{A}$ exists. We write h_x for the linear functional defined by $h_x(a) = h(xa)$ for all $a \in \mathsf{A}$. As a smooth element, x can be written as a finite linear combination

$$x = \sum_{i=1}^{n} \sum_{k,\ell=1}^{n_{s_i}} c(s_i, k, \ell) u_{k\ell}^{(s_i)}.$$

Then the Peter–Weyl orthogonality relations (2.1) imply that $\phi|_{V_s} = h_x|_{V_s} = 0$ for $s \in \mathcal{I}$, $s \notin \{s_1^c, \ldots, s_n^c\}$.

Conversely, if ϕ has finite support, then the sum

$$x = \sum_{s \in \mathcal{I}} \sum_{j,k,\ell=1}^{n_s} D_s \phi(u_{j\ell}^{(s)}) f(u_{kj}^{(s)}) \left(u_{k\ell}^{(s)}\right)^*$$

is finite, therefore $x \in \mathcal{A}$, and the Peter–Weyl orthogonality relations (2.1) imply $\phi|_{\mathcal{A}} = h_x|_{\mathcal{A}}$. Density of $\mathcal{A}$ in A and continuity of ϕ and h_x then give $\phi = h_x$.

Clearly, by the Peter–Weyl orthogonality relations (2.1), x is uniquely determined by $h_x|_{\mathcal{A}} = \phi|_{\mathcal{A}}$. □

Remark 2.6. Note that the density x is uniquely determined by ϕ, even if the Haar state h is not faithful. This is a consequence of the fact that the Haar state is always faithful on the algebra $\mathcal{A}$ of smooth elements.

Denote by $(\pi_h, H, \mathbf{1}_h)$ the GNS representation of A with respect to the Haar state. If the Haar state h is faithful, we can make use of the Tomita–Takesaki theory for Haar states on compact quantum groups [26,28]. Define an antilinear operator S_h on H by

$$S_h \pi_h(a) \mathbf{1}_h = \pi_h(a)^* \mathbf{1}_h,$$

for any $a \in \mathsf{A}$ and set $\Delta_h = S_h^* S_h$. The modular automorphism group $(\sigma_t^h)_{t\in\mathbb{R}}$ is given by

$$\pi_h\big(\sigma_t^h(a)\big)\mathbf{1}_h = \Delta_h^{it} \pi_h(a) \mathbf{1}_h$$

for $a \in \mathcal{A}$. Each element of $\mathcal{A}$ is analytic with respect to the modular group $(\sigma_t^h)_{t\in\mathbb{R}}$.

2.4. *Discrete quantum groups*

Let (A, Δ) be a compact quantum groups. The space of linear functionals on A with finite support has the structure of a discrete algebraic quantum group.

Fix a complete family $(u^{(s)})_{s\in\mathcal{I}}$ of mutually inequivalent irreducible unitary corepresentations of A, and define $e_{k\ell}^{(s)} : \mathcal{A} \to \mathbb{C}$ for $s \in \mathcal{I}$, $1 \le k, \ell \le n_s$ by

$$e_{k\ell}^{(s)}\left(u_{ij}^{(t)}\right) = \delta_{st}\delta_{ki}\delta_{\ell j}$$

for $t \in \mathcal{I}$, $1 \le i, j \le n_s$. These functionals extend to continuous functionals on A, since $e_{k\ell}^{(s)} = h_x$, with $x = D_s \sum_{j=1}^{n_s} f(u_{j\ell}^{(s)}) \left(u_{jk}^{(s)}\right)^*$. The convolution product of two such functionals gives $e_{ij}^{(s)} e_{k\ell}^{(t)} = \delta_{st}\delta_{jk} e_{i\ell}^{(s)}$ for $s, t \in \mathcal{I}$, $1 \le i, j \le n_s$, $1 \le k, \ell \le n_t$, i.e., the linear functionals on A with finite support from a subalgebra

$$\hat{\mathcal{A}} = \operatorname{span}\left\{e_{ij}^{(s)}; s \in \mathcal{I}, 1 \le i, j \le n_s\right\}$$

of A^* with respect to the convolution product. Equip $\hat{\mathcal{A}}$ with the involution $\left(e_{k\ell}^{(s)}\right)^* = e_{\ell k}^{(s)}$. The *-algebra $\hat{\mathcal{A}}$ has the form of a multimatrix algebra,

$$\hat{\mathcal{A}} = \bigoplus_{s\in\mathcal{I}} \operatorname{span}\left\{e_{\ell k}^{(s)}; 1 \le k, \ell \le n_s\right\} \cong \bigoplus_{s\in\mathcal{I}} M_{n_s}(\mathbb{C}) \subseteq \mathsf{A}^*$$

(algebraic direct sum). With the coproduct $\hat{\Delta} : \hat{\mathcal{A}} \to M(\hat{\mathcal{A}} \odot \hat{\mathcal{A}})$ defined by $\hat{\Delta}(\phi)(a \otimes b) = \phi(ab)$ for $a, b \in \mathsf{A}$, $\hat{\mathcal{A}}$ becomes a discrete algebraic quantum group in the sense of [19,20]. Here $M(\hat{\mathcal{A}} \odot \hat{\mathcal{A}})$ denotes the multiplier algebra of $\hat{\mathcal{A}} \odot \hat{\mathcal{A}}$, its elements can be naturally identified with linear functionals on $\mathcal{A} \odot \mathcal{A}$.

For $J \subseteq \mathcal{I}$, we introduce the notation

$$\hat{V}_J = \left\{\phi \in \hat{\mathcal{A}}; \phi\left(u_{ij}^{(s)}\right) = 0 \text{ for } s \notin J, 1 \le i, j \le n_s\right\}$$

for the space of functionals which vanish on the irreducible unitary corepresentations that do not belong to J. We have

$$\hat{V}_{\{s\}} = \mathrm{span}\left\{e^{(s)}_{\ell k};\, 1 \leq k, \ell \leq n_s\right\} \cong M_{n_s}(\mathbb{C})$$

for $s \in \mathcal{I}$ and

$$\hat{V}_{\{s,s^c\}} \cong M_{n_s}(\mathbb{C}) \oplus M_{n_s}(\mathbb{C})$$

if $s \neq s^c$, i.e., if $u^{(s)}$ is not contragredient to itself.

The pair $(\hat{\mathcal{A}}, \hat{\Delta})$ admits an antipode $\hat{S} : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$, which can be characterised by $\big(\hat{S}(\phi)\big)(a) = \phi\big(S(a)\big)$ for $a \in \mathcal{A}, \phi \in \hat{\mathcal{A}}$.

The antilinear map $\mathsf{A}^* \ni \phi \mapsto \phi^\dagger = {}^- \circ \phi \circ * \in \mathsf{A}^*$ allows to characterise the real algebra of hermitian functionals on A as its fixed point algebra. We have $(\phi \star \psi)^\dagger = \phi^\dagger \star \psi^\dagger$ and $(\phi^\dagger)^\dagger = \phi$. Since finitely supported functionals are in the domain of the antipode $\hat{S}$, we have $\phi^\dagger = \hat{S}^{-1}(\phi^*) = \big(\hat{S}(\phi)\big)^*$ for $\phi \in \hat{\mathcal{A}}$.

2.5. First examples

2.5.1. Commutative examples

If G is a compact group, then $\mathsf{A} = C(G)$ becomes a compact quantum group with the coproduct $\Delta : \mathsf{A} = C(G) \to \mathsf{A} \otimes \mathsf{A} \cong C(G \times G)$ defined by

$$\Delta(f)(g_1, g_2) = f(g_1 g_2),$$

for $f \in C(G), g_1, g_2 \in G$. Furthermore, any commutative compact quantum group is of this form, cf. [28, Remark 3 following Definition 1.1].

The Haar state on a commutative compact quantum group $C(G)$ is given by integration against the Haar measure μ of G, i.e., $h(f) = \int_G f \mathrm{d}\mu$ for $f \in C(G)$. It admits a non-trivial square root if and only if the Haar measure μ admits a non-trivial square root. Hence the main theorem of [3] can be reformulated in the following way.

Theorem 2.7 (*[3]*). *Let G be a separable compact group. The pair $(C(G), \Delta)$ admits no non-trivial square root of the Haar state if and only if G is abelian or of the form $H \times E$ where H is the group of unit quaternions and E is a Cartesian product of (at most countably many) copies of $\mathbb{Z}_2$.*

2.5.2. Cocommutative examples

A compact quantum group (A, Δ) is called *cocommutative*, if $\tau \circ \Delta = \Delta$, where $\tau : \mathsf{A} \otimes \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ is the flip, $\tau(a \otimes b) = b \otimes a$. Since irreducible corepresentations of a cocommutative compact quantum group are necessarily one-dimensional, the dense $*$-Hopf algebra of smooth elements in a cocommutative compact quantum group (A, Δ) is of the form $\mathcal{A} = \mathrm{span}(\Gamma)$, where

$$\Gamma = \{u \in \mathsf{A};\, u \text{ unitary and } \Delta(u) = u \otimes u\}$$

is a (discrete) subgroup of the group $\mathcal{U}(\mathsf{A})$ of unitary elements of A.

The Haar state h acts on $u \in \Gamma$ by

$$h(u) = \begin{cases} 1 & \text{if } u = \mathbf{1} \text{ (the trivial corepresentation)}, \\ 0 & \text{otherwise}. \end{cases}$$

If ϕ is a square root of h, then $(\phi \star \phi)(u) = \big(\phi(u)\big)^2 = \delta_{u\mathbf{1}}$ for $u \in \Gamma$, and the only positive square root of the Haar state is the trivial solution $\phi = h$. This argument shows that the Haar state of a cocommutative compact quantum group never admits a non-trivial square root.

2.6. Terminology

Motivated by Theorem 2.7 we introduce the following terminology.

Definition 2.8. A compact quantum group A is said to belong to the quantum *DS*-family if its Haar state does not admit any non-trivial square roots.

We will sometimes refer to a *classical DS-family* as the family of groups listed in Theorem 2.7. The examples discussed above show that one can find and characterise commutative and cocommutative quantum groups in the quantum *DS*-family. A priori it is not clear if there exist at all any elements in the quantum *DS*-family which belong to neither of these classes. We will in fact exhibit such an example in Proposition 6.3.

3. Square roots of the Haar state and nilpotent functionals

We will show in this section that a compact quantum group (A, Δ) is not in the quantum DS-family if and only if there exists a hermitian functional on A that is nilpotent for the convolution, cf. Theorem 3.8. Necessity of this condition is immediate. Indeed, if a state $\phi : \mathsf{A} \to \mathbb{C}$, $\phi \neq h$, is a square root of the Haar state, then $\rho = \phi - h \neq 0$ is hermitian and nilpotent, since

$$\rho \star \rho = \phi \star \phi - \phi \star h - h \star \phi + h \star h = 0.$$

To prove the converse, we follow a similar strategy as Diaconis and Shahshahani. Given a nilpotent hermitian functional ρ, we construct a new ("truncated") hermitian functional ψ_s for which there exists $\epsilon \neq 0$ such that $h + \epsilon\psi_s$ defines a state which is a square root of the Haar state h. But if the Haar state is not a trace, then more care is required to prove the positivity of $h + \epsilon\psi_s$.

We begin with a simple lemma for the tracial case.

Lemma 3.1. *Let A be a unital C^*-algebra with tracial state h. Then we have*

$$|h(xa)| \leq ||x|| h(a)$$

for all $x \in \mathsf{A}$ and $a \in \mathsf{A}_+$.

Proof. Since a is positive, there exists $b \in \mathsf{A}$ such that $a = bb^*$. Denote by π_h and $\mathbf{1}_h$ the GNS representation of h and its cyclic vector representing the state h. Then we have

$$\begin{aligned} |h(xa)| &= |h(xbb^*)| = |h(b^*xb)| = \langle \pi_h(b)\mathbf{1}_h, \pi_h(x)\pi_h(b)\mathbf{1}_h \rangle \\ &\leq ||x|| \langle \pi_h(b)\mathbf{1}_h, \pi_h(b)\mathbf{1}_h \rangle = ||x|| h(b^*b) = ||x|| h(a), \end{aligned}$$

since π_h is a contraction. □

Let us now characterise hermitianity and positivity of a given finitely supported linear functional in terms of its density. Recall that elements in $\mathcal{A}$ are analytic with respect to the modular automorphism group of the Haar state.

Lemma 3.2. *Let (A, Δ) be a compact quantum group with faithful Haar state h and modular group $(\sigma_t^h)_{t\in\mathbb{R}}$, and let $x \in \mathcal{A}$.*

(1) *The functional $h_x \in \mathsf{A}^*$, $h_x(a) = h(xa)$ is hermitian if and only if $\sigma^h_{-i/2}(x)$ is self-adjoint.*
(2) *The functional $h_x \in \mathsf{A}^*$, $h_x(a) = h(xa)$ is positive if and only if $\sigma^h_{-i/2}(x)$ is positive.*

Proof. (1) Denote by A_h^* the space of hermitian continuous functionals on A and once again write π_h and $\mathbf{1}_h$ for the GNS representation of h and its cyclic vector representing the state h. We have

$$\begin{aligned} h_x \in \mathsf{A}_h^* &\Leftrightarrow h(xa^*) = \overline{h(xa)} \quad \forall a \in \mathsf{A} \quad (\text{or } a \in \mathcal{A}) \\ &\Leftrightarrow \langle x^*\mathbf{1}_h, a^*\mathbf{1}_h \rangle = \overline{\langle x^*\mathbf{1}_h, a\mathbf{1}_h \rangle} \quad \forall a \in \mathcal{A} \\ &\Leftrightarrow \langle S_h x\mathbf{1}_h, S_h a\mathbf{1}_h \rangle = \langle a\mathbf{1}_h, x^*\mathbf{1}_h \rangle \quad \forall a \in \mathcal{A} \\ &\Leftrightarrow \langle a\mathbf{1}_h, \underbrace{\Delta_h x\mathbf{1}_h}_{=\sigma^h_{-i}(x)\mathbf{1}_h} \rangle = \langle a\mathbf{1}_h, x^*\mathbf{1}_h \rangle \quad \forall a \in \mathcal{A} \\ &\Leftrightarrow \sigma^h_{-i}(x) = x^* \quad \Leftrightarrow \quad \sigma^h_{-i/2}(x) = \left(\sigma^h_{-i/2}(x)\right)^*, \end{aligned}$$

where we used faithfulness of h and the relation $\sigma^h_{i/2} \circ * = * \circ \sigma^h_{-i/2}$.

(2) Denote by A_+^* the space of positive functionals on A. We have

$$\begin{aligned} h_x \in \mathsf{A}_+^* &\Leftrightarrow h(xbb^*) \geq 0 \quad \forall b \in \mathsf{A} \quad (\text{or } b \in \mathcal{A}) \\ &\Leftrightarrow h\big(\sigma^h_i(b^*)xb\big) \geq 0 \quad \forall b \in \mathcal{A} \\ &\Leftrightarrow \langle \sigma^h_{-i}(b)\mathbf{1}_h, xb\mathbf{1}_h \rangle \geq 0 \quad \forall b \in \mathcal{A}. \end{aligned}$$

Since

$$\begin{aligned} \langle \sigma^h_{-i}(b)\mathbf{1}_h, xb\mathbf{1}_h \rangle &= \langle \Delta_h b\mathbf{1}_h, xb\mathbf{1}_h \rangle = \langle \Delta_h^{1/2} b\mathbf{1}_h, \Delta_h^{1/2} xb\mathbf{1}_h \rangle \\ &= \langle \Delta_h^{1/2} b\mathbf{1}_h, \sigma^h_{-i/2}(x)\Delta_h^{1/2} b\mathbf{1}_h \rangle \end{aligned}$$

and since $\{\Delta_h^{1/2} b\mathbf{1}_h; b \in \mathcal{A}\} = \{\sigma^h_{-i/2}(b)\mathbf{1}_h; b \in \mathcal{A}\}$ is dense, this is equivalent to $\sigma^h_{-i/2}(x) \geq 0$. □

Lemma 3.3. *Let $x \in \mathcal{A}$. If the functional $h_x \in \mathsf{A}^*$ is hermitian, then there exists $\epsilon > 0$ such that $h + \epsilon h_x$ is a positive.*

Proof. Set $\varphi_\epsilon = h + \epsilon h_x = h_{\mathbf{1}+\epsilon x}$. Since $h_x \in \mathsf{A}_h^*$, $\sigma^h_{-i/2}(x)$ is self-adjoint by Lemma 3.2. Therefore there exists $\epsilon > 0$ such that $\mathbf{1} \geq \epsilon\sigma^h_{-i/2}(x) \geq -\mathbf{1}$. Then $\sigma^h_{-i/2}(\mathbf{1}+\epsilon x) = \mathbf{1} + \epsilon\sigma^h_{-i/2}(x) \geq \mathbf{1} - \mathbf{1} = 0$. Since $\mathbf{1} + \epsilon x \in \mathcal{A}$, we can apply Lemma 3.2 and get $\varphi_\epsilon \in \mathsf{A}_+^*$. □

Remark 3.4. Similar methods yield the following general result.

If M is a von Neumann algebra with a faithful normal state ω and $x \in M$ analytic with respect to the modular automorphism group $\{\sigma_t : t \in \mathbb{R}\}$ of the state ω, then

$$|\omega(xa)| \leq \|\sigma_{-\frac{i}{2}}(x)\|\omega(a), \quad \text{for } a \in M_+.$$

Lemma 3.5 (*Truncation Lemma*). *Let $\rho \in \mathsf{A}^*$ be a hermitian functional such that $\rho \star \rho = 0$. For $u^{(s)}$ an irreducible unitary corepresentation of* A*, define ψ_s by*

$$\psi_s\left(u^{(t)}_{k\ell}\right) = \begin{cases} \rho\left(u^{(t)}_{k\ell}\right) & \text{if } t \text{ is equivalent to } s \text{ or } s^c, \\ 0 & \text{else.} \end{cases}$$

Then ψ_s is hermitian, has finite support, and satisfies $\psi_s \star \psi_s = 0$.

Proof. The support of ψ_s is contained in the $*$-closed subspace $V_s + V_{s^c}$ and $\psi_s|_{V_s+V_{s^c}} = \rho|_{V_s+V_{s^c}}$. Therefore ψ_s is clearly hermitian and finitely supported. Furthermore,

$$\begin{aligned}(\psi_s \star \psi_s)(u^{(t)}_{k\ell}) &= \sum_{j=1}^{n_t} \psi_s(u^{(t)}_{kj})\psi_s(u^{(t)}_{j\ell}) \\ &= \begin{cases} \sum_{j=1}^{n_t} \rho(u^{(t)}_{kj})\rho(u^{(t)}_{j\ell}) = (\rho \star \rho)(u^{(t)}_{k\ell}) = 0 & \text{if } t \text{ is equivalent to } s \text{ or } s^c, \\ 0 & \text{else} \end{cases}\end{aligned}$$

for all irreducible unitary corepresentations $u^{(t)}$ of A, i.e., $\psi_s \star \psi_s = 0$. □

Let us first show that it is sufficient to consider compact quantum groups with faithful Haar states.

Lemma 3.6. *Let (A, Δ) be a compact quantum group with not necessarily faithful Haar state h and denote by $(\tilde{\mathsf{A}}, \tilde{\Delta})$ its reduced version, i.e., the compact quantum group with faithful Haar state $\tilde{h}$ obtained from (A, Δ) by dividing out the nullspace of h.*

If $\tilde{h}$ admits a non-trivial square root, then so does h.

Remark 3.7. The converse is also true, and can be shown using truncation arguments similar to those in the proof of Theorem 3.8, but we will not need it.

Proof. Denote by $\tilde{\pi} : \mathsf{A} \to \tilde{\mathsf{A}}$ the canonical projection from (A, Δ) to $(\tilde{\mathsf{A}}, \tilde{\Delta})$, cf. [2]. Then $h = \tilde{h} \circ \tilde{\pi}$ and if $\tilde{h}$ admits a non-trivial square root $\tilde{\phi} \neq h$, then clearly $\phi = \tilde{\phi} \circ \tilde{\pi} \neq h$ defines a non-trivial square root of h. □

Theorem 3.8. *The Haar state h of a compact quantum group (A, Δ) admits a non-trivial square root, i.e., a state $\phi \neq h$ such that $\phi \star \phi = h$, if and only if there exists a bounded non-zero hermitian continuous linear functional on* A *that is nilpotent for the convolution product.*

Proof. If h admits a non-trivial square root ϕ, then clearly $\rho = \phi - h$ defines a bounded non-zero hermitian nilpotent functional.

Conversely, assume that A^* contains a non-zero nilpotent hermitian functional ψ. Then all convolution powers of ψ are also hermitian. If $\psi^{\star n} = 0$, and n is the smallest such number, then set $\rho = \psi^{\star(n-1)}$. This is non-zero and satisfies $\rho \star \rho = 0$. Therefore $\rho(u) = 0$ for any $u \in \mathsf{A}$ with $\Delta(u) = u \otimes u$, in particular $\rho(\mathbf{1}) = 0$.

Since $\rho \neq 0$, there exists an irreducible unitary corepresentation $u^{(s)}$ such that $\rho|_{V_s+V_{s^c}} \neq 0$. Fix such an irreducible unitary corepresentation $u^{(s)}$ and define ψ_s as in the Truncation Lemma (Lemma 3.5). Then ψ_s has finite support and there exists a unique $x \in \mathcal{A}$ such that $\psi_s = h_x \in \mathsf{A}_h^*$, cf. Lemma 2.5.

If h is faithful, then, by Lemma 3.3, there exists $\epsilon > 0$ such that $\phi = h + \epsilon h_x \in \mathsf{A}_+^*$. Since $h \star \psi_s = \psi_s \star h = \psi_s(\mathbf{1})h = 0$, we get

$$\phi \star \phi = h \star h + \epsilon h \star \psi_s + \epsilon \psi_s \star h + \epsilon^2 \psi_s \star \psi_s = h,$$

i.e., ϕ is a non-trivial square root of the Haar state h.

If h is not faithful, then $x \in \mathcal{A}$ can be used to define a nilpotent hermitian functional $\tilde{\psi_s} = \tilde{h}_x$ and a non-trivial square $\tilde{\phi} = \tilde{h} + \epsilon\tilde{h}_x$ on the reduced version $(\tilde{\mathsf{A}}, \tilde{\Delta})$. By Lemma 3.6, $\phi = \tilde{\phi} \circ \tilde{\pi}$ then defines a non-trivial square root of h on (A, Δ). □

Since by the Truncation Lemma we can always choose this nilpotent hermitian linear functional to have finite support, we also get the following characterisation.

Corollary 3.9. *The Haar state h of a compact quantum group (A, Δ) admits a non-trivial square root if and only if its dual discrete algebraic quantum group $(\hat{\mathcal{A}}, \hat{\Delta})$ introduced in Section 2.4 contains a non-zero nilpotent element that is hermitian w.r.t. †.*

We will now show that a compact quantum group whose Haar state has no non-trivial square root is hamiltonian (see Definition 3.11) and that all its quantum subgroups are normal.

Lemma 3.10. *Let* (A, Δ) *be a compact quantum group with faithful Haar state. A quantum subgroup* B *of* A *is normal if and only if the idempotent state* $h_{\mathsf{B}} \circ \pi$ *on* A *induced by the Haar state* h_{B} *of* B *is in the centre of* A^*.

Proof. Denote by

$$E_{\mathsf{A}/\mathsf{B}} = \big(\mathrm{id} \otimes (h_{\mathsf{B}} \circ \pi)\big) \circ \Delta,$$
$$E_{\mathsf{B}\backslash\mathsf{A}} = \big((h_{\mathsf{B}} \circ \pi) \otimes \mathrm{id}\big) \circ \Delta,$$

the conditional expectations onto the coidalgebras A/B and $\mathsf{B}\backslash\mathsf{A}$. The quantum subgroup B is normal if and only these two coidalgebras coincide, cf. Definition 2.4. Since $E_{\mathsf{A}/\mathsf{B}}$ and $E_{\mathsf{B}\backslash\mathsf{A}}$ are unital and preserve the Haar state, by the uniqueness of state preserving conditional expectations this is equivalent to $E_{\mathsf{A}/\mathsf{B}} = E_{\mathsf{B}\backslash\mathsf{A}}$, or

$$f \star (h_{\mathsf{B}} \circ \pi) = f \circ E_{\mathsf{A}/\mathsf{B}} = f \circ E_{\mathsf{B}\backslash\mathsf{A}} = (h_{\mathsf{B}} \circ \pi) \star f$$

for all $f \in \mathsf{A}^*$. □

Definition 3.11. We call a compact quantum group (A, Δ) *hamiltonian* if all idempotent states on A are central in A^* (w.r.t. the convolution).

Idempotent states on finite and compact quantum groups were characterised in [5,6]. For a compact group G, all idempotent states on $C(G)$ are induced by Haar measures of closed subgroups of G, and $C(G)$ is hamiltonian if and only if all closed subgroups of G are normal. Lemma 3.10 shows that all quantum subgroups of hamiltonian compact quantum groups have to be normal. But in general noncommutative compact quantum groups may have idempotent states that are not induced from quantum subgroups.

Proposition 3.12. *Let* (A, Δ) *be a compact quantum group in the quantum DS-family. If the Haar state of* A *is faithful, then* (A, Δ) *is hamiltonian. In particular, every quantum subgroup of* (A, Δ) *is normal.*

Proof. By Theorem 3.8, if h_A admits no non-trivial square root, then A^* contains no hermitian functionals that are nilpotent for the convolution product.

Then the result follows from the fact that in a unital ring without nilpotent elements all idempotents are central, as in [3, Lemma 3]. Since $\mathsf{A}_h^* = \{\phi \in \mathsf{A}^*; \phi \text{ hermitian}\}$ has no nilpotent elements, all idempotent states are central in A_h^*, and therefore also in A^*. □

As mentioned in the introduction the classification of hamiltonian groups plays a very important role in the arguments of [3]. As no such classification is known for (compact) quantum groups, to study compact quantum groups which do not admit nontrivial square roots of Haar states we need to develop other techniques. The next two sections will be devoted to this task.

4. A structure theorem

In this section we characterise compact quantum groups whose Haar state admits no non-trivial square root in terms of their irreducible unitary corepresentations or their dual discrete algebraic quantum group, see Theorem 4.1 and Proposition 4.2.

By Corollary 3.9, we have to characterise discrete algebraic quantum groups which contain no non-zero hermitian nilpotent elements.

For $s \in \mathcal{I}$, we define R_s to be the real algebra of hermitian linear functionals on A that vanish on all irreducible unitary corepresentations of (A, Δ) except s and s^c, i.e.,

$$R_s = \hat{V}_{\{s,s^c\}} \cap \mathsf{A}_h^*.$$

Clearly $R_s = R_{s^c}$. Since $\hat{\mathcal{A}}$ is a multimatrix algebra, the real algebra of all finitely supported hermitian linear functionals $\hat{\mathcal{A}}_h = \hat{\mathcal{A}} \cap \mathsf{A}_h^* =$ decomposes into a direct sum

$$\hat{\mathcal{A}}_h = \bigoplus_{s \in \mathcal{I}_r} R_s, \tag{4.1}$$

where the direct sum runs over the reduced index set $\mathcal{I}_r$ which is obtained from $\mathcal{I}$ by choosing only one representative from each set $\{s, s^c\}$.

Recall that Frobenius showed in [4] that there exist exactly three finite-dimensional division algebras over $\mathbb{R}$, namely the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$, and the skew field of quaternions $\mathbb{H}$.

Theorem 4.1. *A compact quantum group* (A, Δ) *belongs to the quantum DS-family if and only if all summands occurring in the decomposition* (4.1) *are isomorphic to one of the three finite-dimensional division algebras* $\mathbb{R}$, $\mathbb{C}$, *or* $\mathbb{H}$.

Proof. Let us first verify that the conditions are sufficient. If all summands R_s in the direct sum (4.1) are isomorphic to division algebras, then they cannot contain non-zero nilpotent elements, therefore $\hat{\mathcal{A}}_h$ has no non-zero nilpotent elements, either, and by Corollary 3.9 the Haar state of (A, Δ) admits no non-trivial square root.

Conversely, if the Haar state of (A, Δ) has no non-trivial square root, then none of the real algebras R_s occurring in (4.1) contain non-zero nilpotent elements.

Let $s \in \mathcal{I}$. If $u^{(s)}$ is not contragredient to itself, i.e., if $u^{(s)}$ and $u^{(s^c)}$ are not equivalent, then we have $\hat{V}_{\{s\}} \cong M_{n_s}(\mathbb{C})$, $V_s \cap V_{s^c} = \{0\}$, and the map

$$\hat{V}_{\{s\}} \ni \phi \mapsto \phi + \phi^\dagger \in R_s \subseteq \hat{V}_{\{s,s^c\}},$$

where

$$(\phi + \phi^\dagger)(a) = \begin{cases} \phi(a) & \text{if } a \in V_s, \\ \phi^\dagger(a) = \overline{\phi(a^*)} & \text{if } a \in V_{s^c}, \\ 0 & \text{if } a \in V_t \text{ with } t \neq s, s^c, \end{cases}$$

for $\phi \in \hat{V}_{\{s\}}$ is an isomorphism of real algebras. Therefore $R_s \cong \hat{V}_{\{s\}} \cong M_{n_s}(\mathbb{C})$ as real algebras. We see that R_s contains no non-zero nilpotent elements if and only if $n_s = 1$, and in this case $R_s \cong \mathbb{C}$.

Let us now consider the case where $u^{(s)}$ is contragredient to itself. Then we have $V_s = V_{s^c}$, $\hat{V}_{\{s\}} = \hat{V}_{\{s,s^c\}} \cong M_{n_s}(\mathbb{C})$. R_s is a real subalgebra of $\hat{V}_{\{s\}}$ whose complexification coincides with $\hat{V}_{\{s\}}$, since any linear functional on V_s can be written as a complex linear combination of hermitian functionals. $\hat{V}_{\{s\}} \cong M_{n_s}(\mathbb{C})$ is simple, and since the complexification of any real ideal in R_s would be an ideal in $\hat{V}_{\{s\}}$, it follows that R_s is also simple. Therefore, by Wedderburn's theorem ([25], see also [21, Section 13.11]), R_s is isomorphic to a full matrix algebra over a division algebra. In other words, we have $R_s \cong M_m(\mathbb{K})$ for some $m \geq 1$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. If R_s contains no non-zero nilpotent elements, then necessarily $m = 1$, and only the two cases $n_s = 1$ and $R_s \cong \mathbb{R}$, or $n_s = 2$ and $R_s \cong \mathbb{H}$ can occur. □

Let us describe which corepresentations of a compact quantum group (A, Δ) lead to a real algebra of hermitian functionals that is isomorphic to one of the three finite-dimensional division algebras $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.

Let $u = (u_{jk})_{1 \leq j,k \leq n} \in M_n(\mathsf{A})$ be an irreducible unitary corepresentation of a compact quantum group (A, Δ). We denote by $\bar{u} \in M_n(\mathsf{A})$ the corepresentation obtained by taking the adjoints of the coefficients of u, i.e., $\bar{u} = (u^*_{jk})_{1 \leq j,k \leq n}$. The corepresentation $\bar{u}$ is not necessarily unitary, but it is non-degenerate and equivalent to the contragredient corepresentation of u, i.e., there exists an invertible matrix $T \in M_n(\mathbb{C})$ s.t. $\bar{u} = Tu^cT^{-1}$, cf. [28, Proposition 5.2] or [15, Proposition 6.10].

Proposition 4.2. *Let $u \in M_n(\mathsf{A})$ be an irreducible unitary corepresentation of a compact quantum group (A, Δ) and denote by $R(u) \subseteq \mathsf{A}^*_h$ the real algebra given by hermitian linear functionals on A which vanish on the coefficients of all irreducible unitary corepresentations that are not equivalent to u or u^c. Then we have the following characterisations of $R(u)$.*

(i) *$R(u) \cong \mathbb{R}$ if and only if u is one-dimensional and contragredient to itself. This is the case if and only if u is unitary, self-adjoint, and group-like (i.e., $\Delta(u) = u \otimes u$).*
(ii) *$R(u) \cong \mathbb{C}$ if and only if u is one-dimensional and not contragredient to itself. This is the case if and only if u is unitary and group-like, but not self-adjoint.*
(iii) *$R(u) \cong \mathbb{H}$ if and only if u is two-dimensional and there exists an invertible matrix $Q \in M_2(\mathbb{C})$ such that $\bar{u} = TuT^{-1}$, where $T = \bar{Q}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}Q^{-1}$. In particular, this implies that u is contragredient to itself.*

Proof. The first two cases follow from the proof of Theorem 4.1 if we note that a one-dimensional unitary corepresentation is contragredient to itself if and only if it is self-adjoint.

Let us now prove (iii). The real division algebra of quaternions can be realised as

$$\mathbb{H} = \left\{ \begin{pmatrix} \alpha + i\beta & i\gamma - \delta \\ i\gamma + \delta & \alpha - i\beta \end{pmatrix} : \alpha, \beta, \gamma, \delta \in \mathbb{R} \right\}.$$

Its complexification is isomorphic to $M_2(\mathbb{C})$, and the elements of $\mathbb{H}$ can be characterised in $M_2(\mathbb{C})$ as the hermitian elements for the anti-linear homomorphism $\dagger : M_2(\mathbb{C}) \to M_2(\mathbb{C})$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix}.$$

Dualising these relations we see that the real algebra $R(u)$ associated to a unitary corepresentation u is isomorphic to $\mathbb{H}$ if and only if u is two-dimensional and if the subspace $V(u)$ spanned by the coefficients of u admits a basis $a_{11}, a_{12}, a_{21}, a_{22}$ such that

$$a^*_{11} = a_{22}, \qquad a^*_{12} = -a_{21}, \quad \text{and} \quad \Delta(a_{jk}) = \sum_{\ell=1}^{2} a_{j\ell} \otimes a_{\ell k} \quad \text{for } 1 \leq j, k \leq 2.$$

Therefore $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C})$ is a corepresentation of (A, Δ). It is non-degenerate, since its coefficients form a basis of $V(u)$, so by Maes and Van Daele [15, Proposition 6.4] it is equivalent to a unitary corepresentation, which we can choose to be u. I.e. there exists an invertible matrix $Q \in M_2(\mathbb{C})$ s.t. $u = QaQ^{-1}$. We get

$$\bar{u} = \bar{Q}\bar{a}\bar{Q}^{-1} = \bar{Q}FaF^{-1}\bar{Q}^{-1} = TuT^{-1}$$

with $F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \bar{Q}FQ^{-1}$. □

Note that the characterisation above seems to be new even for standard compact groups, together with Theorem 2.7 yielding the following corollary.

Corollary 4.3. *Let G be a separable compact group. The following conditions are equivalent:*

(i) *G admits only one-dimensional and two-dimensional irreducible representations, each two-dimensional irreducible representation $U : G \to M_2(\mathbb{C})$ is self-contragredient and such that there exists an invertible matrix $Q \in M_2(\mathbb{C})$ such that $\bar{U} = TUT^{-1}$, where $T = \bar{Q}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}Q^{-1}$;*

(ii) *$G \approx H \times E$ where H is the group of unit quaternions and E is a Cartesian product of (at most countably many) copies of $\mathbb{Z}_2$.*

Let us now consider the Woronowicz quantum group $SU_q(2)$. This example will play an important role in the next section, when we show that a compact quantum group whose Haar state admits no non-trivial square root is necessarily of Kac type, i.e., its Haar state is a trace, cf. Theorem 5.1.

Example 4.4. Let $q \in \mathbb{R}\backslash\{0\}$. Denote by $SU_q(2) = (\mathsf{A}, \Delta)$ the Woronowicz quantum group introduced in [27], i.e., the universal C^*-algebra generated by the four generators $u_{11}, u_{12}, u_{21}, u_{22}$ with the coproduct determined by $\Delta u_{jk} = \sum_{\ell=1}^2 u_{j\ell} \otimes u_{\ell k}$ for $j, k = 1, 2$, and the $*$-algebraic relations $uu^* = I = u^*u$ and $\bar{u} = F_q u F_q^{-1}$ with $u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ and $F_q = \begin{pmatrix} 0 & q \\ -1 & 0 \end{pmatrix}$, i.e.,

$$\sum_{\ell=1}^{2} u_{j\ell}u_{k\ell}^* = \delta_{jk} = \sum_{\ell=1}^{2} u_{\ell j}^* u_{\ell k},$$
$$u_{11}^* = u_{22}, \qquad u_{12}^* = -qu_{21}.$$

Note that $SU_q(2)$ is isomorphic to the universal orthogonal quantum group $A_o(\tilde{F}_q)$ defined by Van Daele and Wang [22], where $\tilde{F}_q$ is given by

$$\tilde{F}_q = \frac{F_q}{\sqrt{|\det F_q|}} = \begin{pmatrix} 0 & \mathrm{sign}(q)\sqrt{|q|} \\ -\frac{1}{\sqrt{|q|}} & 0 \end{pmatrix}.$$

The irreducible unitary corepresentations have been determined in [26,27,23,14,11]. For each non-negative half-integer $s \in \frac{1}{2}\mathbb{Z}_+$ there exists a $2s+1$-dimensional irreducible unitary corepresentation $u^{(s)} = (u_{k\ell}^{(s)})_{1\le k,\ell\le 2s+1}$ of $SU_q(2)$, which is unique up to unitary equivalence and contragredient to itself. The map $\phi \to \phi^\dagger$ on the dual discrete algebraic quantum group maps the summands $\hat{V}_{\{s\}} \cong M_{2s+1}(\mathbb{C})$ in the decomposition (4.1) to themselves and takes the form

$$A^\dagger = Q\overline{A}Q^{-1}$$

where $A \mapsto \overline{A}$ denotes entry-wise complex conjugation and

$$Q = \left((-1)^j q^{j-1}\delta_{j,n-k+1}\right)_{1\le j,k\le 2s+1} \in M_{2s+1}(\mathbb{C}).$$

For example for $s = \frac{1}{2}$, the fundamental corepresentation $u^{(1/2)} = u$, $Q = F_q$, and

$$R_{1/2} \cong \left\{ \begin{pmatrix} a & -q\overline{b} \\ b & \overline{a} \end{pmatrix}; a, b \in \mathbb{C} \right\}.$$

For $q > 0$, $\mathbb{H} \ni \alpha + \beta I + \gamma J + \delta K \mapsto \begin{pmatrix} \alpha + i\beta & \sqrt{q}(-\gamma + i\delta) \\ \frac{1}{\sqrt{q}}(\gamma + i\delta) & \alpha - i\beta \end{pmatrix} \in R_{1/2}$ defines an isomorphism of real algebras and $R_{1/2}$ contains no non-zero nilpotent elements.

For $q < 0$, $R_{1/2}$ is isomorphic to $M_2(\mathbb{R})$ and contains nilpotent elements, e.g., $\begin{pmatrix} \sqrt{q} & -q \\ 1 & -\sqrt{q} \end{pmatrix}$.

The higher dimensional irreducible unitary corepresentations always give non-zero nilpotent hermitian functionals.

5. Kac property and the 'local' structure of quantum groups in the quantum *DS*-family

As an application of Theorem 4.1 we will now show that compact quantum groups in the quantum *DS*-family are necessarily *of Kac type*, i.e., its Haar state is a trace. Recall that the Haar state of a compact quantum group (A, Δ) is tracial if and only if the antipode S on $\mathcal{A}$ is involutive, see [28, Theorem 1.5]. This is the case if and only if for any unitary corepresentation $u = (u_{jk})_{1\leq j,k\leq n}$ the corepresentation $\bar{u} = (u^*_{jk})_{1\leq j,k\leq n}$ obtained by taking adjoints component-wise is again unitary.

Hence being of Kac type is in a sense a 'local' property, which will be clear from the proof of Theorem 5.1. We first need to recall a few more facts and definitions.

A compact quantum group (A, Δ) is called a *compact matrix quantum group* if it has a finite-dimensional corepresentation u whose coefficients generate A as a C^*-algebra. It follows from [15, Proposition 3.7] that the C^*-algebra $\mathsf{A}(u) = C^*\left(\{u_{jk} : 1 \leq j, k \leq n\}\right)$ generated by the coefficients of any unitary corepresentation $u \in M_n(\mathbb{C})$ of a compact quantum group (A, Δ) is a compact matrix quantum group with the restriction of the coproduct of A. We will call $(\mathsf{A}(u), \Delta|_{\mathsf{A}(u)})$ the *quotient quantum group* of (A, Δ) induced by u. Equivalent corepresentations clearly induce isomorphic quotient quantum groups.

Theorem 5.1. *Let* (A, Δ) *be a compact quantum group in the quantum DS-family. Then* (A, Δ) *is of Kac type.*

Proof. Assume that (A, Δ) is in the quantum *DS*-family. It is sufficient to show that the square of the antipode acts identically on the coefficients of the irreducible unitary corepresentations of (A, Δ). By Theorem 4.1, the irreducible unitary corepresentations of (A, Δ) have dimension one or two.

Let us first consider the one-dimensional corepresentations. If a is the coefficient of a one-dimensional unitary corepresentation of (A, Δ), then a is group-like, i.e., $\varepsilon(a) = 1$ and $\Delta(a) = a \otimes a$. Therefore $S(a) = a^* = a^{-1}$ and $S^2(a) = a$.

Let now $u = (u_{jk})_{1\leq j,k\leq 2} \in M_2(\mathsf{A})$ be a two-dimensional irreducible unitary corepresentation of (A, Δ). We will show that the quotient quantum group $(\mathsf{A}(u), \Delta|_{\mathsf{A}(u)})$ is a quantum subgroup of $SU_q(2)$ for some $0 < q \leq 1$.

By Theorem 4.1 and Proposition 4.2, there exists an invertible matrix $Q \in M_2(\mathbb{C})$ such that

$$\overline{u} = TuT^{-1},$$

with $T = \bar{Q}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}Q^{-1}$. The matrix T satisfies the relation $T\overline{T} = -I$, so by Bichon et al. [1, Equation (5.4)], there exist $0 < q \leq 1$ and a unitary matrix $U \in M_2(\mathbb{C})$ such that

$$T = U^t\begin{pmatrix} 0 & q \\ -\frac{1}{q} & 0 \end{pmatrix}U.$$

Let $v = UuU^*$, then clearly v is a two-dimensional irreducible unitary corepresentation of (A, Δ) and $(\mathsf{A}(v), \Delta|_{\mathsf{A}(v)}) = (\mathsf{A}(u), \Delta|_{\mathsf{A}(u)})$. Furthermore, v satisfies the relation

$$\begin{aligned}
\overline{v} &= \overline{UuU^*} = \overline{U}\overline{u}U^t = \overline{U}TuT^{-1}U^t \\
&= \overline{U}U^t\begin{pmatrix} 0 & q \\ -\frac{1}{q} & 0 \end{pmatrix}UuU^{-1}\begin{pmatrix} 0 & q \\ -\frac{1}{q} & 0 \end{pmatrix}^{-1}(U^t)^{-1}U^t \\
&= \begin{pmatrix} 0 & q \\ -\frac{1}{q} & 0 \end{pmatrix}v\begin{pmatrix} 0 & q \\ -\frac{1}{q} & 0 \end{pmatrix}^{-1},
\end{aligned}$$

i.e., the coefficients of v satisfy the defining relations of the universal orthogonal quantum group $A_0\begin{pmatrix} 0 & q \\ -\frac{1}{q} & 0 \end{pmatrix} \cong SU_{q^2}(2)$, cf. [22] or Example 4.4. If we denote by $w = (w_{jk})_{1\leq j,k\leq 2}$ the generators of $SU_{q^2}(2)$, then $w_{jk} \mapsto v_{jk}$ defines a surjective morphism of compact quantum groups from $SU_q(2)$ to $(\mathsf{A}(v), \Delta|_{\mathsf{A}(v)})$, i.e., $(\mathsf{A}(v), \Delta|_{\mathsf{A}(v)})$ is a quantum subgroup of $SU_{q^2}(2)$.

In Example 4.4 we have seen that the Haar state on $SU_{q^2}(2)$ admits a non-trivial square root, so $(\mathsf{A}(u), \Delta|_{\mathsf{A}(u)})$ has to be a proper quantum subgroup of $SU_{q^2}(2)$. For $q^2 \neq 1$, the quantum subgroups of $SU_{q^2}(2)$ are the torus $\mathbb{T}$ and its subgroups, cf. [17] or also [7]. These are classical groups and therefore of Kac type. For $q^2 = 1$ we get $SU_1(2)$, which is a also classical group and of Kac type.

It follows that the quotient quantum group $(\mathsf{A}(u), \Delta|_{\mathsf{A}(u)})$ induced by any irreducible unitary corepresentation u is of Kac type, therefore we have $S^2 = \mathrm{id}$ on the dense $*$-Hopf algebra contained in A, and (A, Δ) is of Kac type. □

The proof of the above theorem shows that the structure of quotient quantum groups induced by two-dimensional irreducible corepresentations is in fact quite rigid. This is formalised in the following corollary.

Corollary 5.2. *A compact quantum group* (A, Δ) *belongs to the quantum DS-family if and only if the following two conditions are satisfied:*

(i) (A, Δ) *admits only one- and two-dimensional irreducible unitary corepresentations;*
(ii) *the quotient quantum groups* $(\mathsf{A}(u), \Delta|_{\mathsf{A}(u)})$ *of* (A, Δ) *induced by its two-dimensional irreducible unitary corepresentations are isomorphic to* $C(H)$, *where* H *is the eight-element group of unit quaternions.*

Proof. Let (A, Δ) belong to the quantum *DS*-family. Condition (i) follows from Theorem 4.1 and Proposition 4.2. Let then u be a two-dimensional irreducible unitary corepresentation of (A, Δ). The second part of the proof of Theorem 5.1 implies that $(\mathsf{A}(u), \Delta|_{\mathsf{A}(u)})$ is isomorphic to a proper quantum subgroup of $C(SU_q(2))$ for some $q \in (0, 1]$, so, due to [17], also to a proper quantum subgroup of $C(SU(2))$.

Condition (ii) then follows by inspection of the subgroups of $SU(2)$ (see, e.g., [17]), since H is the only subgroup of $SU(2)$ which is in the *DS*-family and which has a two-dimensional irreducible unitary representation.

Conversely, if (A, Δ) satisfies (i)–(ii) above, then each of its irreducible unitary corepresentations verifies one of the conditions in Proposition 4.2. Thus Theorem 4.1 ends the proof. □

The above corollary can be interpreted as describing the 'local' structure of the elements of the quantum *DS*-family – we know that they only admit one- and two-dimensional irreducible corepresentations and have now the full understanding of what types of quotient quantum groups are generated by each of the irreducible corepresentations (for one-dimensional corepresentations the resulting quotients are just the algebras of functions on cyclic groups). It has one other important consequence. Recall the Nichols–Zoeller theorem that states that the dimension of a finite-dimensional Hopf algebra is divisible by the dimensions of its Hopf subalgebras, cf. [16]. Together with the above corollary it implies the following result.

Corollary 5.3. *Let* (A, Δ) *be a non-cocommutative compact quantum group in the quantum DS-family. If* A *is finite-dimensional, then its dimension is divisible by eight.*

We finish this section by describing the structure of the algebra of functions on H in greater detail; this will be of use in the next section.

Example 5.4. Denote by $\pm 1, \pm I, \pm J, \pm K$ the eight unit quaternions, with the relations $I^2 = J^2 = K^2 = -1, I \cdot J = K$, $J \cdot I = -K$, etc. Denote by λ_g and $\mathbf{1}_{\{g\}}, g \in \{\pm 1, \pm I, \pm J, \pm K\}$ the corresponding bases for $C^*(H) \cong \mathbb{C}H$ and $C(H)$. Besides the constant function $\mathbf{1}_H$, H has three more one-dimensional irreducible unitary representations σ_I, σ_J and σ_K, which are uniquely determined by

$$\begin{aligned} &\sigma_I(I) = 1, \qquad \sigma_I(J) = -1, \\ &\sigma_J(I) = -1, \qquad \sigma_J(J) = 1, \\ &\sigma_K(I) = -1, \qquad \sigma_K(J) = -1. \end{aligned}$$

Furthermore, H has, up to unitary equivalence, a unique two-dimensional irreducible unitary representation $\pi : H \to M_2(\mathbb{C})$ of H (or, equivalently, corepresentation of $C(H)$), given by

$$\pi(I) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \pi(J) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Denote by $\pi_{jk} \in C(H)$, $1 \leq j, k \leq 2$ the matrix elements of H w.r.t. to the standard basis of $\mathbb{C}^2$. Then $\{\mathbf{1}_H, \sigma_I, \sigma_J, \sigma_K, \pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}\}$ is a basis of $C(H)$. We set

$$C(H)_0 = \operatorname{span}\{\mathbf{1}_H, \sigma_I, \sigma_J, \sigma_K\} \quad \text{and} \quad C(H)_1 = \operatorname{span}\{\pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}\}.$$

We have $\Delta\pi_{jk} = \sum_{\ell=1}^2 \pi_{j\ell} \otimes \pi_{\ell k}$ and $\Delta g = g \otimes g$ for the one-dimensional unitary corepresentation of $C(H)$. Furthermore, on can check that the tensor product of the two-dimensional representation of H decomposes into a direct sum of the four one-dimensional representations. From these observations it follows that the decomposition $C(H) = C(H)_0 \oplus C(H)_1$ defines a $\mathbb{Z}_2$-grading of $C(H)$, i.e., we have

$$\begin{aligned} \big(C(H)_j\big)^* &\subseteq C(H)_j, \\ C(H)_j \cdot C(H)_k &\subseteq C(H)_{j+k}, \\ \Delta C(H)_j &\subseteq C(H)_j \otimes C(H)_j \end{aligned}$$

for $j, k \in \mathbb{Z}_2$. Equivalently, the map $d : C(H) \to C(H) \otimes \mathbb{C}\mathbb{Z}_2$ defined by

$$d(u) = u_0 \otimes \delta_0 + u_1 \otimes \delta_1$$

for $u = u_0 + u_1$ with $u_0 \in \mathsf{A}_0, u_1 \in \mathsf{A}_1$, and $\delta_g, g \in \mathbb{Z}_2$ the standard basis of $C^*(\mathbb{Z}_2) \cong \mathbb{C}\mathbb{Z}_2$, defines a coaction of $C^*(\mathbb{Z}_2)$ on $C(H)$.

In yet another way we can interpret the grading described above as resulting from the fact that $\mathbb{C}\mathbb{Z}_2$ is a quotient Hopf *-algebra of $C(H)$; this will be used in Example 6.2.

6. Combining the 'local' structure of the compact quantum groups in the quantum *DS*-family into the global one and genuinely quantum examples

In the last section we showed that if (A, Δ) is in the quantum *DS*-family, then we can completely determine the quotient quantum subgroups induced by individual irreducible corepresentations of A. Here we show that they can be combined in a non-trivial way to provide the examples which belong to the quantum *DS*-family and are neither commutative nor cocommutative.

It is easy to see that the quotient quantum subgroup induced by an arbitrary number of one-dimensional irreducible corepresentations of a compact quantum group is always cocommutative, so of the form $C^*(\Gamma)$ for some (discrete) group Γ. We therefore begin our analysis by combining a one-dimensional corepresentation with a two-dimensional one.

Proposition 6.1. *Let (A, Δ) be in the quantum DS-family. Let $u \in M_2(\mathsf{A})$ be a two-dimensional irreducible unitary corepresentation and $g \in \mathsf{A}$ a one-dimensional unitary corepresentation. Then there exists a unitary matrix $U \in M_2(\mathbb{C})$ s.t.*

$$\begin{pmatrix} gu_{11} & gu_{12} \\ gu_{21} & gu_{22} \end{pmatrix} = U \begin{pmatrix} u_{11}g^{-1} & u_{12}g^{-1} \\ u_{21}g^{-1} & u_{22}g^{-1} \end{pmatrix} U^*.$$

Proof. Multiplying u by g, we get a two-dimensional irreducible unitary corepresentation $gu = \begin{pmatrix} gu_{11} & gu_{12} \\ gu_{21} & gu_{22} \end{pmatrix}$. By Proposition 4.2, u and ug are contragredient to themselves. Since (A, Δ) is Kac, $\bar{u}$ and $\overline{gu} = \bar{u}g^{-1}$ are unitary. Therefore the pairs u and $\bar{u}$, and gu and $\bar{u}g^{-1}$, are unitarily equivalent, which implies that gu and ug^{-1} are also unitarily equivalent. □

Let u be a two-dimensional irreducible unitary corepresentation of a compact quantum group (A, Δ) in the quantum *DS*-family, and let $g_1, \ldots, g_n$ be one-dimensional unitary corepresentations that do not belong to $\mathsf{A}(u)$. The above proposition suggests that $(\mathsf{A}(u), \Delta|_{\mathsf{A}(u)})$ and $(\mathsf{A}(g_1 \oplus \cdots \oplus g_n), \Delta|_{\mathsf{A}(g_1 \oplus \cdots \oplus g_n)})$ are a matched pair, and that quotient quantum group $(\mathsf{A}(v), \Delta|_{\mathsf{A}(v)})$ generated by the direct sum $v = u \oplus g_1 \oplus \cdots \oplus g_n$ is given by a bicrossproduct of $(\mathsf{A}(u), \Delta|_{\mathsf{A}(u)})$ and $(\mathsf{A}(g_1 \oplus \cdots \oplus g_n), \Delta|_{\mathsf{A}(g_1 \oplus \cdots \oplus g_n)})$. We use this idea to construct examples of noncommutative, noncocommutative compact quantum groups of Kac type whose Haar state admits no non-trivial square roots.

Example 6.2. Let Γ be a commutative discrete group and $C(H) = C(H)_0 \oplus C(H)_1$ the algebra of functions on the eight-element group of unit quaternions, with the $\mathbb{Z}_2$-grading introduced in Example 5.4. Denote the quotient Hopf *-algebra morphism from $C(H)$ onto $\mathbb{C}\mathbb{Z}_2$ by p. Note that $\mathbb{Z}_2$ acts on $\mathbb{C}\Gamma$ via the standard (period 2) automorphism, mapping γ to γ^{-1} and let $\tilde{\alpha} : \mathbb{C}\mathbb{Z}_2 \otimes \mathbb{C}\Gamma \to \mathbb{C}\Gamma$ denote the corresponding map on the level of Hopf *-algebras, so that

$$\tilde{\alpha}(\lambda_0 \otimes \gamma) = \lambda_0 \otimes \gamma, \qquad \tilde{\alpha}(\lambda_1 \otimes \gamma) = \lambda_1 \otimes \gamma^{-1}.$$

Define now the action $\alpha : C(H) \otimes \mathbb{C}\Gamma \to \mathbb{C}\Gamma$ by the formula

$$\alpha = \tilde{\alpha} \circ (p \otimes \mathrm{id}_{\mathbb{C}\Gamma}).$$

It turns $\mathbb{C}\Gamma$ into a left $C(H)$-module algebra and comodule algebra, as follows from the fact that $\tilde{\alpha}$ has this property and p is a Hopf algebra morphism.

Consider now the crossed (smashed) product construction of $\mathbb{C}\Gamma \rtimes C(H)$, following for example [13, Proposition 1.6.6]. The vector space $K = \mathbb{C}\Gamma \odot C(H)$ can be turned into an algebra with the multiplication (we use here and below the Sweedler notation for coproducts)

$$(a \otimes u)(b \otimes v) = a\alpha(u_{(1)} \otimes b) \otimes u_{(2)}v$$

($u, v \in C(H), a, b \in \mathbb{C}\Gamma$). Moreover the proof of [13, Proposition 6.2.1] implies that with the standard tensor coproduct

$$\Delta_K(a \otimes u) = a_{(1)} \otimes u_{(1)} \otimes a_{(2)} \otimes u_{(2)},$$

K becomes a Hopf algebra – note that although $C(H)$ is not cocommutative, so formally we cannot apply directly [13, Proposition 6.2.1], in fact the action α is defined so that it only 'sees' the cocommutative quotient $\mathbb{C}\mathbb{Z}_2$. This explains why we still obtain the desired result.

Using further the definition of α we get

$$(a \otimes u_j)(b \otimes v) = a\varepsilon(u_{j(1)})S^j(b) \otimes u_{j(2)}v = aS^j(b) \otimes u_jv$$

for $j \in \{0, 1\}, u_j \in C(H)_j, v \in C(H), a, b \in \mathbb{C}\Gamma$. One can check that K becomes further a $*$-Hopf algebra with

$$(a \otimes u_j)^* = \big((a \otimes 1)(1 \otimes u_j)\big)^* = (1 \otimes u_j^*)(a^* \otimes 1) = S^j(a^*) \otimes u_j^*$$

for $j \in \{0, 1\}, u_j \in C(H)_j, a \in \mathbb{C}\Gamma$. Since Haar states are invariant under the respective antipodes, we see the tensor product of the Haar states on $C^*(\Gamma)$ and $C(H)$ defines a normalised positive integral on K, i.e., K is an algebraic compact quantum group in the sense of [19,20].

If $u \in M_n\big(C(H)\big))$ is an irreducible unitary corepresentation of $C(H)$ and $g \in \Gamma$, then $u \otimes g$ is unitary in K, e.g., in the simplified notation,

$$(g \otimes u)(g \otimes u)^* = (g \otimes u)(S(g^*) \otimes u^*) = (g \otimes u)(g \otimes u^*) = gS(g) \otimes uu^* = 1 \otimes 1.$$

Therefore, and since the coproduct in K is simply the tensor product of the coproducts of $\mathbb{C}\Gamma$ and $C(H)$, we see that the irreducible unitary corepresentations of K are of the form $g \otimes u$, with u an irreducible unitary corepresentation of $C(H)$, and $g \in \Gamma$. Since their coefficients span K, we can deduce that K is the $*$-Hopf algebra of a compact quantum group, for which we can choose the universal C^*-algebra of the $*$-algebra K. Let us denote this quantum group by $C^*\Gamma \rtimes_\alpha C(H)$.

The construction above can also be viewed as a special case of the double crossed product construction from [12, Proposition 3.12] or [13, Example 6.2.12].

It is clear that $C^*\Gamma \rtimes_\alpha C(H)$ has only one- and two-dimensional irreducible unitary corepresentations, and straightforward to show that the two-dimensional irreducible corepresentations are self-contragredient and satisfy the condition in Proposition 4.2. Hence the Haar state of $C^*\Gamma \rtimes_\alpha C(H)$ does not admit any non-trivial square roots.

Proposition 6.3. *Let Γ be an abelian discrete group which contains elements that are not of order two. The crossed product $C^*\Gamma \rtimes_\alpha C(H)$ constructed in Example 6.2 is a noncommutative, noncocommutative compact quantum group in the quantum DS-family.*

Proposition 6.1 describes the interaction of a two-dimensional irreducible corepresentation, say u, of a quantum group in the quantum DS-family with one-dimensional ones. It is reflected by certain equivalences between $u \otimes g$ and $g^{-1} \otimes u$. In the last part of the paper we discuss certain aspects of the interaction between different two-dimensional representations. To this end we need to introduce a certain equivalence relation on the equivalence classes of irreducible corepresentations of a fixed compact quantum group. Introduce first the notation: if (A, Δ) is a compact quantum group, let $\mathrm{Irr}(\mathsf{A})$ denote the set of the equivalence classes of irreducible corepresentations of A and let $\Gamma_{\mathsf{A}} \subset \mathrm{Irr}(\mathsf{A})$ denote the equivalence classes of one-dimensional corepresentations (in other words, group-like elements of A). It is well-known (and has been used above) that the tensor product of corepresentations provides Γ_{A} with the structure of a discrete group.

Proposition 6.4. *Let (A, Δ) be a compact quantum group. The relation $\approx_\Gamma$ on $\mathrm{Irr}(\mathsf{A})$ given by the formula*

$$u \approx_\Gamma v \quad \text{if } \exists_{\gamma \in \Gamma_{\mathsf{A}}} \, u = v \otimes \gamma$$

is an equivalence relation.

Proof. Easy check, essentially a consequence of the fact that Γ_{A} forms a group and associativity of the tensor operation for (not necessarily irreducible) corepresentations of A. □

The set of equivalence classes in $\mathrm{Irr}(\mathsf{A})$ with respect to the relation $\approx_\Gamma$ will be denoted $\mathrm{Irr}(\mathsf{A})/\approx_\Gamma$ and for $u \in \mathrm{Irr}(\mathsf{A})$ the corresponding equivalence class $\mathrm{Irr}(\mathsf{A})/\approx_\Gamma$ will be denoted $[u]_\approx$. Note that all one-dimensional corepresentations form an equivalence class with respect to the relation $\approx_\Gamma$, to be denoted $[1]_\approx$.

Theorem 6.5. *Let (A, Δ) be in the quantum DS-family. Then the set $\mathrm{Irr}(\mathsf{A})/\approx_\Gamma$ is equipped with a well-defined product, which is given by the following condition: for $u, v, w \in \mathrm{Irr}(\mathsf{A})$*

$$[u]_\approx \cdot [v]_\approx = [w]_\approx \quad \text{if } \exists_{\gamma \in \Gamma_{\mathsf{A}}} \, u \otimes v \succeq w \otimes \gamma. \tag{6.1}$$

Moreover the pair $(\mathrm{Irr}(\mathsf{A})/\approx_\Gamma, \cdot)$ forms an abelian group, in which each non-trivial element has order 2.

Proof. We need to check first that the product in the formula (6.1) is well defined. If $\gamma, \gamma' \in \mathrm{Irr}(\mathsf{A})$ are one-dimensional, then so is $\gamma \otimes \gamma'$ and according to the notation of (6.1) and that introduced before the statement of the theorem we have $[1]_\approx \cdot [1]_\approx = [1]_\approx$. If $u \in \mathrm{Irr}(\mathsf{A})$ is two-dimensional and $\gamma \in \Gamma_{\mathsf{A}}$, then $u \otimes \gamma$ is irreducible and equivalent to u with respect to $\approx_\Gamma$, so $[u]_\approx \cdot [1]_\approx = [u]_\approx$. Similarly $\gamma \otimes u$ is irreducible. Moreover, as both u and $\gamma \otimes u$ are self-contragredient by Proposition 4.2, $\gamma \otimes u = u \otimes \gamma^{-1}$ (recall that the equality here is understood in terms of the usual equivalence classes of irreducible representations).

The only non-trivial case is that of $u, v \in \mathrm{Irr}(\mathsf{A})$ both two-dimensional. Observe that due to Corollary 5.2 and the discussion in Example 5.4 the tensor product $u \otimes u$ is a four-dimensional corepresentation decomposing into 4 one-dimensional corepresentations, including the trivial one. We will distinguish two-possibilities: first assume that $u \approx_\Gamma v$. Then there is some $\gamma \in \Gamma_{\mathsf{A}}$ such that $u = v \otimes \gamma = \gamma^{-1} \otimes v$. Hence $u \otimes v = \gamma^{-1} \otimes (v \otimes v)$ is a direct sum of four one-dimensional corepresentations (trivially equivalent to each other with respect to $\approx_\Gamma$). This can be rephrased by writing $[u]_\approx \cdot [u]_\approx = [1]_\approx$. It remains to consider the possibility of $u \not\approx_\Gamma v$. Then $u \otimes v$ is a four-dimensional, necessarily reducible corepresentation. Suppose that $u \otimes v$ contains a one-dimensional corepresentation, say γ. Then $(\gamma^{-1} \otimes u) \otimes v$ contains a trivial corepresentation, and as both $\gamma^{-1} \otimes u$ and v are irreducible, this would mean that $\gamma^{-1} \otimes u = v^c = v$, so $u \approx_\Gamma v$ – contradiction. Thus $u \otimes v$ decomposes into a direct sum of two two-dimensional irreducible corepresentations, say w and w'. To assure that the product in (6.1) is well defined we need to prove that $w \approx_\Gamma w'$ (strictly speaking we also need to show that $[w]_\gamma$ depends only on the $\approx_\Gamma$-equivalence classes of u and v, but this is easy to see). Tensor the corepresentation $u \otimes v$ on the left with u. Then it decomposes into four two-dimensional irreducible corepresentations, each $\approx_\Gamma$-equivalent to v. Hence in particular $u \otimes w$ as a subrepresentation of $u \otimes v$ decomposes into two-dimensional irreducible corepresentations

$\approx_\Gamma$-equivalent to v, say $\gamma_1 \otimes v$, and $\gamma_2 \otimes v$. Tensor the formula $u \otimes w = (\gamma_1 \otimes v) \oplus (\gamma_2 \otimes v)$ again on the left with u. Then on the left we obtain the direct sum of four two-dimensional corepresentations, each $\approx_\Gamma$-equivalent to w, and on the right the direct sum of four two-dimensional corepresentations, two of which are $\approx_\Gamma$-equivalent to w, and two are $\approx_\Gamma$-equivalent to w'. Hence w and w' are $\approx_\Gamma$-equivalent and the proof of the main part of the theorem is finished.

As to the fact that the product $\cdot$ gives $\mathrm{Irr}(\mathsf{A})/\approx_\Gamma$ the group structure described in the theorem, it suffices to observe that $\cdot$ inherits associativity from the usual associativity of tensor products of corepresentations and that the first part of the proof shows that $[1]_\approx$ is the neutral element for $\cdot$ and each element in $\mathrm{Irr}(\mathsf{A})/\approx_\Gamma$ is its own inverse. □

The above theorem implies that if (A, Δ) is in the quantum *DS*-family then $(\mathrm{Irr}(\mathsf{A})/\approx_\Gamma, \cdot)$ is a direct sum of (possibly infinitely many) copies of $\mathbb{Z}_2$. For cocommutative (A, Δ) the group $(\mathrm{Irr}(\mathsf{A})/\approx_\Gamma, \cdot)$ is trivial. For a compact group G in the *DS*-family the group $(\mathrm{Irr}(C(G))/\approx_\Gamma, \cdot)$ is either trivial or a two-element group, depending on whether G contains the group of unit quaternions. If (A, Δ) is the compact quantum group constructed in Example 6.2, then again the group $(\mathrm{Irr}(\mathsf{A})/\approx_\Gamma, \cdot)$ is isomorphic to $\mathbb{Z}_2$. It is therefore natural to seek the answer to the following open question.

Problem 6.6. Does there exist (A, Δ) in the quantum *DS*-family such that the group $(\mathrm{Irr}(\mathsf{A})/\approx_\Gamma, \cdot)$ has more than two elements?

It is natural to seek for such an object among finite-dimensional Kac algebras, exploiting the existing classification of low-dimensional examples (see [8]). Corollary 5.3 implies that the dimension of such a Kac algebra would have to be divisible by 8. To allow for two distinct classes of two-dimensional irreducible corepresentations we need the dimension to be at least 16. The case by case analysis of the form of the Grothendieck rings of 16-dimensional Kac algebras listed in [10] implies that none of these algebras can provide a positive answer to the question asked in Problem 6.6. Hence the lowest dimension for the Kac algebra that would answer the question in Problem 6.6 is equal 24.

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